Carnot-Carathéodory metric and gauge fluctuations in Noncommutative Geometry

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Abstract

[*] Gauge fields have a natural metric interpretation in terms of horizontal distance. This distance, also called Carnot-Carathéodory or sub-Riemannian distance, is by definition the length of the shortest horizontal path between points, that is to say the shortest path whose tangent vector is everywhere horizontal with respect to the gauge connection. In noncommutative geometry all the metric information is encoded within the Dirac operator D. In the classical case, i.e. commutative, Connes's distance formula allows one to extract from D the geodesic distance on a Riemannian spin manifold. In the case of a gauge theory with a gauge field A, the geometry of the associated $U(n)$-vector bundle is described by the covariant Dirac operator $D+A$. What is the distance encoded within this operator ? It was expected that the noncommutative geometry distance d defined by a covariant Dirac operator was intimately linked to the Carnot-Carathéodory distance d_H defined by A. In this paper we make this link precise, showing that the equality of d and d_H strongly depends on the holonomy of the connection. Quite interestingly we exhibit an elementary example, based on a 2-torus, in which the noncommutative distance has a very simple expression and simultaneously avoids the main drawbacks of the Riemannian metric (no discontinuity of the derivative of the distance function at the cut-locus) and of the sub-Riemannian one (memory of the structure of the fiber).

I Introduction

Noncommutative geometry[2] enlarges differential geometry beyond the scope of Riemannian spin manifolds and gives access, as shown in various examples, to spaces obtained as the product of the continuum by the discrete. It allows one to describe in a single and coherent geometrical object the space-time of the Standard Model of elementary particles [†] coupled with Euclidean general relativity[1]. Specifically, the diffeomorphism group of general relativity appears as the automorphism group of $C^\infty(M)$, the algebra of smooth functions over a compact Riemannian spin manifold M, while the gauge group of the strong and electroweak interactions emerges as the group $U(\mathcal{A}_I)$ of unitary elements of a finite dimensional algebra $\mathcal{A}_I$ (modulo a lift to the spinors[11]). Remarkably, unitaries not

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[†]with massless neutrinos. Massive Dirac neutrinos are easily incorporated in the model[16] as long as one of them remain massless. Otherwise more substantial changes might be required.

only act as gauge transformations but also acquire a metric significance via the so-called *fluctuations of the metric*. This paper aims to study in detail the analogy introduced in [3] between a simple kind of fluctuations of the metric, those governed by a connection 1-form on a principal bundle, and the associated Carnot-Carathéodory metric.

A noncommutative geometry consists in a *spectral triple*

$$\mathcal{A},\ \mathcal{H},\ D$$

where $\mathcal{A}$ is an involutive algebra, commutative or not, $\mathcal{H}$ a Hilbert space carrying a representation Π of $\mathcal{A}$ and D a selfadjoint operator on $\mathcal{H}$. Together with a chirality operator Γ and a real structure J both acting on $\mathcal{H}$, they satisfy a set of properties[3] providing the necessary and sufficient conditions for 1) an axiomatic definition of Riemannian spin geometry in terms of commutative algebra 2) its natural extension to the noncommutative framework. Points are recovered as pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$, in analogy with the commutative case where

$$\mathcal{P}(C^\infty(M)) \simeq M \tag{1}$$

$$\omega_x(f) = f(x) \tag{2}$$

for any pure state ω_x and smooth function f. A distance d between states ω, ω' of $\mathcal{A}$ is defined by

$$d(\omega,\omega') \doteq \sup_{a\in\mathcal{A}} \left\{ |\omega(a)-\omega'(a)|\,;\ \|[D,\Pi(a)]\| \leq 1 \right\} \tag{3}$$

where the norm is the operator norm on $\mathcal{H}$. In the commutative case,

$$\mathcal{A}_E = C^\infty(M)\,,\ \mathcal{H}_E = L_2(M,S),\ D_E = -i\gamma^\mu\partial_\mu \tag{4}$$

with $\mathcal{H}_E$ the space of square integrable spinors and D_E the ordinary Dirac operator of quantum field theory, d coincides with the geodesic distance defined by the Riemannian structure of M. Thus (3) is a natural extension of the classical distance formula, all the more as it does not involve any notion ill-defined in a quantum framework such as the trajectory between points.

Carnot-Carathéodory metrics (or sub-Riemannian metrics)[15] are defined on manifolds P equipped with a *horizontal distribution*, that is to say a (smooth) specification at any point $p\in P$ of a subspace H_pP of the tangent space T_pP. The Carnot-Carathéodory distance d_H between p and q is the length of the shortest path c joining p and q whose tangent vector is everywhere horizontal,

$$d_H(p,q) = \operatorname*{Inf}_{\dot{c}(t)\in H_{c(t)}P} \int_0^1 \|\dot{c}(t)\|\, dt. \tag{5}$$

If there is no horizontal path from p to q then $d_H(p,q)$ is infinite. Any point at finite distance from p is said *accessible*

$$\mathrm{Acc}(p) \doteq \{q\in P;\ d_H(p,q) < +\infty\}. \tag{6}$$

Most often the norm in the integrand of (5) comes from an inner product in the horizontal subspace. The latter can be obtained in (at least) two ways: either by restricting to HP a Riemannian structure of TP or, when $P \xrightarrow{\pi} M$ is a fiber bundle with a connection, by

pulling back the Riemannian structure g of M. In the latter case the horizontal distribution is the kernel of the connection 1-form and any horizontal vector has norm

$$\|u\| \doteq \|\pi_*(u)\| = \sqrt{g(\pi_*(u), \pi_*(u))}. \tag{7}$$

Note that (5) provides P with a distance although P may not be a metric manifold, only M is asked to be Riemannian.

By taking the product of a Riemannian geometry (4) by a spectral triple with finite dimensional $\mathcal{A}_I$, one obtains as pure state space a $U(\mathcal{A}_I)$-bundle P over M. A connection on P then not only defines a Carnot-Carathéodory distance d_H but also, via the process of fluctuation of the metric recalled in section II, a distance d similar to (3) except that the ordinary Dirac operator D is replaced by the covariant differentiation operator associated to the connection-1 form. In section III we compare the connected components for these two distances: while a connected component for d_H is also connected for d, a connected component of d is not necessarily connected for d_H. We investigate the importance of the holonomy group on this matter. In section IV we show that the two distances coincide when the holonomy is trivial. In the non-trivial case we work out some necessary conditions on the holonomy group that may allow d to equal d_H. In section V we treat in detail a simple low-dimensional example in which each of the connected components of d_H is a dense subset of a two dimensional torus $\mathbb{T}$. As a main result of this paper we show in section VI that while the Carnot-Carathéodory metric forgets about the fiber bundle structure of $\mathbb{T}$, the noncommutative metric deforms it in a quite intriguing way: from a specific intrinsic point of view, the fiber acquires the shape of a cardioid. Hence the classical 2-torus inherits a metric which is "truly" noncommutative in the sense that it cannot be described in (sub)Riemannian nor discrete terms. This is, to our knowledge, the novelty of the present work.

Notations and conventions:

- M is a Riemannian compact spin manifold of dimension m without boundary. Cartesian coordinates are labeled by Greek indices μ, ν and we use Einstein summation over repeated indices in alternate positions (up/down).

- $\mathcal{P}(\mathcal{A})$ denotes the set of pure states of $\mathcal{A}$ (positive, linear applications from $\mathcal{A}$ to $\mathbb{C}$, with norm 1 and that do not decompose as a convex combination of other states). Throughout this paper we deal with a pure state space which is a trivial bundle P over M, with fiber $\mathbb{C}P^{n-1}$. An element of P is written ξ_x where x is a point of M and $\xi \in \mathbb{C}P^{n-1}$.

- Most of the time we omit the symbol Π and it should be clear from the context whether a means an element of $\mathcal{A}$ or its representation on $\mathcal{H}$. Unless otherwise specified a bracket denotes the scalar product on $\mathbb{C}^n$.

- We use the result of [6] according to which the supremum in (3) can be sought on positive elements of $\mathcal{A}$.

II Fluctuations of the metric

In noncommutative geometry a connection on a geometry $(\mathcal{A}, \mathcal{H}, D)$ is defined via the identification of $\mathcal{A}$ as a finite projective module over itself (i.e. as the noncommutative equivalent of the sections of a vector bundle via the Serre-Swan theorem)[3]. It is implemented by replacing D with a *covariant operator*

$$D_A \doteq D + A + JAJ^{-1} \tag{8}$$

where J is the real structure and A is a selfadjoint element of the set Ω^1 of 1-forms

$$\Omega^1 \doteq \left\{ a^i[D, b_i] \; ; \; a^i, b_i \in \mathcal{A} \right\}. \tag{9}$$

Only the part of D_A that does not obviously commute with the representation, namely

$$\mathcal{D} \doteq D + A, \tag{10}$$

enters in the distance formula (3) and induces a so-called *fluctuation of the metric*. In the following we consider almost commutative geometries obtained as the product of the continuous - external - geometry (4) by an internal geometry $\mathcal{A}_I$, $\mathcal{H}_I$, D_I. The product of two spectral triples, defined as

$$\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_I, \; \mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_I, \; D = D_E \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \tag{11}$$

where $\mathbb{I}_I$ is the identity operator of $\mathcal{H}_I$ and γ^5 the chirality of the external geometry, is again a spectral triple. The corresponding 1-forms are[8,17]

$$-i\gamma^\mu f^i_\mu \otimes m_i + \gamma^5 h^j \otimes n_j$$

where $m_i \in \mathcal{A}_I$, $h^j, f^i_\mu \in C^\infty(M)$, $n_j \in \Omega^1_I$. Selfadjoint 1-forms decompose into an $\mathcal{A}_I$-valued skew-adjoint 1-form field over M, $A_\mu \doteq f^i_\mu m_i$, and an Ω^1_I-valued selfadjoint scalar field $H \doteq h^j n_j$.

When the internal algebra $\mathcal{A}_I$ has finite dimension, A_μ takes values in the Lie algebra of unitaries of $\mathcal{A}$ and is called the *gauge part* of the fluctuation. In [14] we have computed the noncommutative distance (3) for a scalar fluctuation only ($A_\mu = 0$). In [3] the distance is considered for a pure gauge fluctuation ($H = 0$) obtained from the internal geometry

$$\mathcal{A}_I = M_n(\mathbb{C}), \quad \mathcal{H}_I = M_n(\mathbb{C}), \quad D_I = 0,$$

that is to say

$$\mathcal{D} = -i\gamma^\mu(\partial_\mu \otimes \mathbb{I}_I + \mathbb{I}_E \otimes A_\mu). \tag{12}$$

$\mathcal{A}_E$ being nuclear, the set of pure states of

$$\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C}) = C^\infty(M, M_n(\mathbb{C})) \tag{13}$$

is[7] $\mathcal{P}(\mathcal{A}) \simeq \mathcal{P}(\mathcal{A}_E) \times \mathcal{P}(\mathcal{A}_I)$, where $\mathcal{P}(\mathcal{A}_I)$ is the projective space $\mathbb{C}P^{n-1}$,

$$\omega_\xi(m) = \langle \xi, m\xi \rangle = \mathrm{Tr}(s_\xi \, m) \tag{14}$$

for $m \in \mathcal{A}_I, \xi \in \mathbb{C}P^{n-1}$ and s_ξ the support of ω_ξ. The evaluation of $\xi_x \doteq (\omega_x, \omega_\xi)$ on $a = f^i \otimes m_i \in \mathcal{A}$ reads

$$\xi_x(a) = \mathrm{Tr}(s_\xi \, a(x)) \tag{15}$$

where

$$a(x) \doteq f^i(x) \otimes m_i. \tag{16}$$

Hence $\mathcal{P}(\mathcal{A})$ is a trivial bundle

$$P \xrightarrow{\pi} M$$

with fibre $\mathbb{C}P^{n-1}$.

The gauge potential A_μ defines both a horizontal distribution H on P, with associated Carnot-Carathéodory metric d_H, and a noncommutative metric d given by formula (3) with $\mathcal{D}$ substituted for D. In the case of a zero connection, $\mathcal{D} = D_E$ and d is the geodesic distance on M. Indeed the commutator norm condition $\|[D_E, f]\| \leq 1$ forces the gradient of f to be smaller than 1, so that

$$d(\omega_x, \omega_y) = \sup_{\|\mathrm{grad}\, f\| \leq 1} |f(x) - f(y)| \leq \int_0^1 \|\dot{c}(t)\|\, dt = d_{\mathrm{geo}}(x, y) \tag{17}$$

where c, $c(0) = x$, $c(1) = y$ is a minimal geodesic from x to y. One then easily checks that this upper bound is attained by the function

$$L(z) \doteq d_{\mathrm{geo}}(x, z) \quad \forall z \in M \tag{18}$$

(or more precisely by a sequence of smooth functions converging to the continuous function L). As we shall see in the following section, in the case of a non-zero connection, one obtains without difficulty a result similar to (17) with d_H playing the role of d_{geo} (cf eq. (19) below). However, except in some simple cases studied in section IV, d_H is not the least upper bound and there is no simple equivalent to the function L. In fact the main part of this paper, especially section V, is devoted to building the element $a \in \mathcal{A}$ that realizes the supremum in the distance formula.

III Connected components

We say that two pure states ξ_x, ζ_y are connected for d if and only if $d(\xi_x, \zeta_y)$ is finite.

Proposition III.1 *For any ξ_x in P, $Acc(\xi_x)$ is connected for d.*

Proof. The result is obtained by showing that for any $\zeta_y \in \mathrm{Acc}(\xi_x)$,

$$d(\xi_x, \zeta_y) \leq d_H(\xi_x, \zeta_y). \tag{19}$$

Let us start by recalling how to transfer the covariant derivative[9] of a section V of P,

$$\nabla_\mu V = \partial_\mu V + A_\mu V,$$

to the algebra $\mathcal{A}$. Given $a \in \mathcal{A}$, the evaluation (15) is the diagonal of the sesquilinear form defined fiberwise on the vector bundle $P' \xrightarrow{\pi'} M$ with fiber $\mathbb{C}^n$,

$$(W'_x, V'_x) \mapsto \langle W'_x, a(x) V'_x \rangle \tag{20}$$

for $W'_x, V'_x \in \pi'^{-1}(x)$. Accordingly, as a $C^\infty(M)$-module, we view $\mathcal{A}$ as the sections of the bundle P'' of rank-two tensors on M

$$a = a_{ij}\, \overline{e^i} \otimes e^j$$

with values in $\overline{T^*\mathbb{C}^n} \otimes T^*\mathbb{C}^n$. Here $\{e^i\}$ is the dual of the canonical basis $\{e_i\}$ of $T\mathbb{C}^n \simeq \mathbb{C}^n$ and $\{\overline{e^i}\}$ its complex conjugate

$$\overline{e^i}(V) = \overline{V^i} \;\text{ for }\; V = V^i e_i \in \mathbb{C}^n.$$

The covariant derivative of P then naturally extends to P'' [‡]

$$\nabla_\mu a = \partial_\mu a + [A_\mu, a]. \tag{21}$$

Let us fix a horizontal curve of pure states $c(t)$, $t \in [0,1]$, between ξ_x and ζ_y as defined in (15). Let (π, V) be a trivialization in P such that

$$\pi(\xi_x) = x,\;\; V(\xi_x) = \xi \qquad \pi(\zeta_y) = y,\;\; V(\zeta_y) = \zeta \tag{22}$$

and define

$$V(t) \doteq V(c(t)).$$

c is the horizontal lift starting at ξ_x of the curve

$$c_*(t) \doteq \pi(c(t))$$

lying in M and tangent to

$$\pi_*(\dot{c}) = \dot{c}_* = \dot{c}_*^\mu \partial_\mu. \tag{23}$$

Writing $s(t)$ for the support of the pure state $\omega_{V(t)}$, the curve $t \mapsto s(t)$ is horizontal in P'' in the sense of the covariant derivative (21) [§]

$$\nabla_{\dot{c}_*} s \doteq \dot{c}_*^\mu \,\nabla_\mu s = 0. \tag{24}$$

Let us associates to any $a \in \mathcal{A}$ its evaluation f along c,

$$f(t) \doteq \mathrm{Tr}(s(t) a(c_*(t))), \tag{25}$$

whose derivative with respect to t is easily computed using (24)

$$\dot{f} = \mathrm{Tr}(s \,\nabla_{\dot{c}_*} a). \tag{26}$$

At a given t the Cauchy-Schwarz inequality yields the bound

$$|\dot{f}(t)| \leq \left\| df_{|t} \right\| \|\dot{c}_*(t)\| \tag{27}$$

where df is the 1-form on c_* with components

$$\partial_\mu f = \mathrm{Tr}(s \nabla_\mu a). \tag{28}$$

$s[\mathcal{D}, a]s$ evaluated at some $c_*(t)$ is an $n' \times n$ square matrix ($n' = \dim \mathcal{H}_E$ is the dimension of the spin representation),

$$s[\mathcal{D}, a]s = -i\gamma^\mu \otimes s(\nabla_\mu a)s = -i\gamma^\mu \partial_\mu f \otimes s, \tag{29}$$

[‡] $\left\{ \begin{array}{l} \nabla_\mu e^i = -A^i_{\mu k} e^k \\ \nabla_\mu \overline{e^i} = -\overline{A^i_{\mu k}}\, \overline{e^k} \end{array} \right.$ hence $\nabla_\mu a \doteq \nabla_\mu (a_{ij}\, \overline{e^i}) \otimes e^j + a_{ij}\, \overline{e^i} \otimes \nabla_\mu e^j = (\partial_\mu a_{ij} + [A, a]_{ij}) \overline{e^i} \otimes e^j$.

[§] in Dirac notation c horizontal in P is written $|\dot{V}\rangle + \dot{c}^\mu A_\mu |V\rangle = 0$. By simple manipulations $\left\{ \begin{array}{l} |\dot{V}\rangle\langle V| + \dot{c}^\mu A_\mu |V\rangle\langle V| = 0 \\ |V\rangle\langle \dot{V}| - |V\rangle\langle V| \dot{c}^\mu A_\mu = 0 \end{array} \right.$, hence $\dot{s} = |V\rangle\langle\dot{V}| + |\dot{V}\rangle\langle V| = \dot{c}^\mu [|V\rangle\langle V|, A_\mu] = \dot{c}^\mu [s, A_\mu]$.

with norm $\|df_{|t}\|$. Therefore

$$\|df_{|t}\| \leq \sup_{x\in M} \|[\mathcal{D},a]_{|x}\| = \|[\mathcal{D},a]\| \tag{30}$$

so, as soon as $\|[\mathcal{D},a]\| \leq 1$,

$$|\xi_x(a) - \zeta_y(a)| = |\int_0^1 \dot{f}(t)\,dt\,| \leq \int_0^1 \|\dot{c}_*(t)\|\,dt, \tag{31}$$

which precisely means $d(\xi_x, \zeta_y) \leq d_H(\xi_x, \zeta_y)$. ■

It would be tempting to postulate that d and d_H have the same connected components. Half of this way is done in the proposition above. The other half would consist in checking that d is infinite as soon as d_H is infinite. However this is, in general, not the case. It seems that there is no simple conclusion on that matter since we shall exhibit in section V an example in which some states that are not in $\mathrm{Acc}(\xi_x)$ are at finite noncommutative distance from ξ_x whereas others are at infinite distance. The best we can do for the moment is to work out (Proposition III.3 below) a sufficient condition on the holonomy group associated to the connection A_μ that guarantees the non-finiteness of $d(\xi_x, \zeta_y)$ for $\zeta_y \notin \mathrm{Acc}(\xi_x)$. We begin with the following elementary lemma.

Lemme III.2 *$d(\xi_x, \zeta_y)$ is infinite if and only if there is a sequence $a_n \in \mathcal{A}$ such that*

$$\mathop{lim}_{n\to+\infty} \|[D,a_n]\| \to 0, \qquad \mathop{lim}_{n\to+\infty} |\xi_x(a_n) - \zeta_y(a_n)| = +\infty. \tag{32}$$

Proof. The point is to show that from a sequence a_n satisfying

$$\|[D,a_n]\| \leq 1\ \forall n \in \mathbb{N}, \qquad \lim_{n\to+\infty} |\xi_x(a_n) - \zeta_y(a_n)| = +\infty$$

one can extract a sequence $\tilde{a}_n$ satisfying (32). This is done by considering

$$\tilde{a}_n \doteq \frac{a_n}{\sqrt{|\xi_x(a_n) - \zeta_y(a_n)|}}.$$

■

Proposition III.3 *Let $\xi, \zeta \in \mathbb{C}P^{n-1}$. If there exists a matrix $M \in M_n(\mathbb{C})$ that commutes with the holonomy group at x, $Hol(x)$, and such that*

$$Tr(s_\xi M) \neq Tr(s_\zeta M), \tag{33}$$

then $d(\omega, \omega') = +\infty$ for any $\omega \in Acc(\xi_x)$, $\omega' \in Acc(\zeta_x)$.

Proof. The proof is a restatement of a classical result (cf [12] p.113) according to which an element of $\mathcal{A}$ invariant under the adjoint action of the holonomy group is a parallel tensor, that is to say $\nabla_\mu a = 0$ in all directions μ. We detail this point in the following for the sake of completeness.

From now on we fix a trivialization (π, V) on $P = \mathcal{P}(\mathcal{A})$. Recall that given a curve from $c_*(0) = x$ to $c_*(1) = y \in M$, the end point of the horizontal lift c of c_* with initial condition $c(0) = (x, \xi)$ is $c(1) = (y, U_{c_*}(1)\xi)$ where

$$U_{c_*}(t) = P\exp(-\int_{c_*(t)} A_\mu dx^\mu)$$

(P is the time-ordered product) is the solution of

$$\dot{U} = -\dot{c}^{\mu} A_{\mu} U. \tag{34}$$

In the following we write U_{c_*} for $U_{c_*}(1)$. Let $M \in M_n(\mathbb{C})$ commute with $\mathrm{Hol}(x)$. Define $a_M \in \mathcal{A}$ by

$$a_M(x) \doteq M$$

and for any $y \in M$,

$$a_M(y) \doteq U_{c_*} a_M(x) U_{c_*}^* \tag{35}$$

where c_* is a curve joining x to y. One checks that $a_M(y)$ commutes with any $V_l \in \mathrm{Hol}(y)$ since

$$V_l\, a_M(y) V_l^* = U_{c_*} U_{c_*}^* V_l U_{c_*} a_M(x) U_{c_*}^* V_l^* U_{c_*} U_{c_*}^* = a_M(y)$$

where we use that $U_{c_*}^* V_l U_{c_*}$ belongs to $\mathrm{Hol}(x)$. Hence (35) uniquely defines $a_M(y)$ since parallel transporting $a_M(x)$ along another curve c'_* yields

$$a'_M(y) = U_{c'_*} U_{c_*}^* a_M(y) U_{c_*} U_{c'_*}^* = a_M(y)$$

where we used that $U_{c_*} U_{c'_*}^* \in \mathrm{Hol}(y)$. Using (34) one explicitly checks that

$$\nabla_{\dot{c}_*} a_M = 0.$$

Since this is true for any curve c_*, a_M is parallel so

$$[\mathcal{D}, a_M] = 0.$$

Now (33) means that $\xi_x(a_M) - \zeta_x(a_M) \neq 0$, hence $d(\xi_x, \zeta_x) = +\infty$ by lemma III.2, and the result follows by the triangle inequality. ■

Proposition above only provides sufficient conditions. Whether they are necessary, i.e. whether from $d(\xi_x, \zeta_y) = +\infty$ on can build a matrix M that commutes with the holonomy group and do not cancel the difference of the states is an open question. Lemma III.2 suggests that to any infinite distance is associated a tensor that commutes with the Dirac operator. Moreover it is not difficult to show that any parallel tensor commutes with the holonomy group. Therefore the question is: are the parallel tensors the only ones that commute with D ? For the time being the answer is not clear to the author.

To close this section, let us mention a situation in which the two metrics have the same connected components.

Corollary III.4 *If for a given $\xi_x \in P$ the vector space*

$$\mathcal{H}_{hol} \doteq Span\{U\xi\,;\, U \in Hol(x)\}$$

has dimension $h < n$, then $Acc(\xi_x)$ is the connected component of ξ_x for d.

Proof. In an orthonormal basis $\{\mathcal{B}_{hol}, \mathcal{B}\}$ of $\mathbb{C}^n$ with $\mathcal{B}_{hol}$ a basis of $\mathcal{H}_{hol}$, $\mathrm{Hol}(x)$ is block represented, so

$$M = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{n-h} \end{pmatrix}$$

commutes with $\mathrm{Hol}(x)$. Moreover $\mathrm{Tr}(s_\xi M) = 0$. On the contrary for any $\zeta_x \notin \mathrm{Acc}(\xi_x)$, the rank one projector s_ζ does not project on $\mathcal{H}_{hol}$ so $\mathrm{Tr}(s_\zeta M) \neq 0$. Therefore, by Proposition III.3, $d(\xi_x, \zeta_y)$ is infinite for any $\zeta_y \notin \mathrm{Acc}(\xi_x)$, hence the result by Proposition III.1. ■

IV Flat case versus holonomy constraints

The preceding section suggests that the two metrics defined by a connection on the pure state space P of the algebra (13), the Carnot-Carathéodory distance d_H and the noncommutative distance d, do not coincide. It is likely that the two metrics do not have the same connected components as soon as the conditions of Proposition III.3 are not fulfilled. However nothing forbids d from equalling d_H on each connected component of d. We already know that $d \leq d_H$ so to obtain the equality it would be enough to exhibit one positive $a \in \mathcal{A}$ (or a sequence of elements a_n) satisfying the commutator norm condition as well as

$$\xi_x(a) - \zeta_y(a) = d_H(\xi_x, \zeta_y). \tag{36}$$

The existence of such an a strongly depends on the holonomy of the connection: when the latter is trivial, e.g. by the Ambrose-Singer theorem when the connection is flat and M simply connected, then the two metrics are equal, as shown below in Proposition IV.1. When the holonomy is non-trivial, we work out in Proposition IV.4 some necessary conditions on the shortest path that may forbid d from equalling d_H.

Proposition IV.1 *When the holonomy group reduces to the identity, $d = d_H$ on all P.*

Proof. For $\zeta_y \notin \mathrm{Acc}(\xi_x)$, Corollary III.4 yields

$$d(\xi_x, \zeta_y) = +\infty = d_H(\xi_x, \zeta_y).$$

Thus we focus on the case $\zeta_y \in \mathrm{Acc}(\xi_x)$. By Cartan's structure equation the horizontal distribution defined by a connection with trivial holonomy is involutive, which means that the set of horizontal vector fields is a Lie algebra for the Lie bracket inherited from TP. Equivalently (Frobenius theorem) the bundle of horizontal vector fields is integrable. Hence $\mathrm{Acc}(\xi_x)$ is a submanifold of P, call it Ξ, such that $Tp\Xi = H_pP$ for any $p \in \Xi$. For any $z \in M$ there is at least 1 point in the intersection

$$\pi^{-1}(z) \cap \Xi$$

(e.g. the end point of the horizontal lift, starting at ξ_x, of any curve from x to z) and only one (otherwise there would be a horizontal curve joining two distinct points in the fiber, contradicting the triviality of the holonomy). In other words all the horizontal lifts starting at ξ_x of curves joining x to z have the same end point, call it $\sigma(z)$, and the application

$$\sigma : z \mapsto \pi^{-1}(z) \cap \Xi$$

defines a smooth section of P. Hence

$$\Xi = \sigma(M).$$

Note that $\zeta_y = \sigma(y)$ is the only point in the fiber over y which is at finite distance from $\xi_x = \sigma(x)$. Considering the horizontal lift of the Riemannian geodesic from x to y, it turns out that d_H on Ξ coincides with the geodesic distance d_{geo} on M. The sequence of elements a_n we are looking for in (36) is a sequence approximating the continuous $M_2(\mathbb{C})$-valued function

$$L \otimes \mathbb{I} \tag{37}$$

where L is the geodesic distance function (18). ■

The difficulty arises when the shortest horizontal curve c does not lie in a horizontal section. This certainly happens when the connection is not flat and/or M not simply connected. As soon as the holonomy is non-trivial, different points ξ_x, ζ_x on the same fiber can be at finite non-zero Carnot-Carathéodory distance from one another although the Riemannian distance of their projections vanishes. The question reduces to finding the equivalent of the element (37) in the closure of $\mathcal{A}$ that attains the supremum in (36). A natural candidate to play the role of the function L in the case of a non-trivial holonomy is the fiber-distance function which associates to any $z \in M$ the length of the shortest horizontal path joining ξ_x to some point in $\pi^{-1}(z)$. When the holonomy is trivial this function precisely coincides with L. However there is no natural candidate to play the role of the identity matrix in (37). Possibly one might determine by purely algebraic techniques which element a of $\mathcal{A}$ realizes the supremum in the distance formula. The best approach we found for the moment is to work out, Proposition IV.4, some conditions between the matrix part of a and the self-intersecting points of c_* that are necessary for d to equal d_H.

Definition IV.2 *Given a curve c in a fiber bundle with horizontal distribution H, we call a c-ordered sequence of K self-intersecting points at p_0 a set of at least two elements $\{c(t_0), c(t_1), ..., c(t_K)\}$ such that*

$$\pi(c(t_i)) = \pi(c(t_0)), \quad d_H(c(t_0), c(t_i)) > d_H(c(t_0), c(t_i))$$

for any $i = 1, ..., K$ (Figure 1).

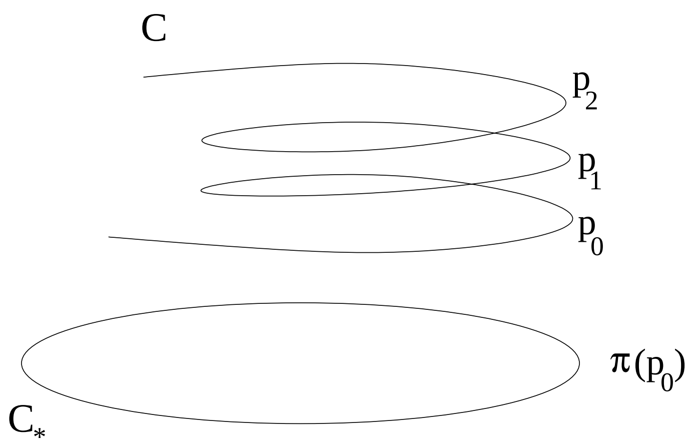

Figure 1: An ordered sequence of self-intersecting points, with $p_i = c(t_i)$.

Lemme IV.3 *Let ξ_x, ζ_y be two points in P such that $d(\xi_x, \zeta_y) = d_H(\xi_x, \zeta_y)$. Then for any $c(t)$ belonging to a minimal horizontal curve c between $c(0) = \xi_x$ and $c(1) = \zeta_y$,*

$$d(\xi_x, c(t)) = d_H(\xi_x, c(t)). \tag{38}$$

Moreover, for any such curve there exists an element $a \in \mathcal{A}$ (or a sequence a_n) such that

$$\xi_t(a) = d_H(\xi_x, c(t)) \tag{39}$$

for any $t \in [0, 1]$, where ξ_t denotes $c(t)$ viewed as a pure state of $\mathcal{A}$.

Proof. We write the proof assuming that the supremum in the distance formula is attained by some $a \in \mathcal{A}$. In case the supremum is not reached, the proof is identical using a sequence $\{a_n\}$. Assume a does satisfies the commutator norm condition as well as (36). Let us parameterize c by its length element τ and use "dot" for the derivative $\frac{d}{d\tau}$. The function $f(t) = \xi_t(a)$ defined by (25) has constant derivative along c_*. Indeed (36) reads

$$\int_0^\Lambda \dot{f}(\tau) d\tau = \Lambda \tag{40}$$

where $\Lambda = d_H(\xi_x, \zeta_y)$. Since $\|\dot{c}_*(\tau)\| = 1$ for any $\tau \in [0, \Lambda]$, (27) and (30) forbid $|\dot{f}(\tau)|$ from being greater than 1. Hence

$$\dot{f}(\tau) = 1 \tag{41}$$

for almost every τ. Thus for any $\lambda \leq \Lambda$,

$$\int_0^\lambda \dot{f}(\tau) d\tau = \lambda \tag{42}$$

which reads

$$\xi_\lambda(a) - \xi_x(a) = \lambda = d_H(\xi_x, \xi_\lambda). \tag{43}$$

Hence (38) by Proposition III.1, and (39) by considering $\tilde{a} \doteq a - \xi_x(a)$. ■

Applying lemma IV.3 to the self-intersecting points defined in IV.2 one obtains the announced necessary conditions for d to equal d_H.

Proposition IV.4 *The noncommutative distance between two points ξ_x, ζ_y in P can equal the Carnot-Carathéodory one only if there exists a minimal horizontal curve c between ξ_x and ζ_y such that there exists an element $a \in \mathcal{A}$, or a sequence of elements a_n, satisfying the commutator norm condition as well as*

$$\xi_{t_i}(a) = d_H(\xi_x, c(t_i)) \quad or \quad \lim_{n\to\infty} \xi_{t_i}(a_n) = d_H(\xi_x, c(t_i)) \tag{44}$$

for any $\xi_{t_i} = c(t_i)$ in any c-ordered sequence of self-intersecting points.

Given a sequence of K self-intersecting points at p, Proposition IV.4 puts $K + 1$ condition on the n^2 real components of the selfadjoint matrix $a(\pi(p))$. So it is most likely that a necessary condition for $d(\xi_x, \zeta_y)$ to equal $d_H(\xi_x, \zeta_y)$ is the existence of a minimal horizontal curve between ξ_x and ζ_y such that its projection does not self-intersect more than $n^2 - 1$ times. We will refine this interpretation in the example of the next section. From a more general point of view it is not clear how to deal with such a condition in the framework of sub-Riemannian geometry¶. It might be possible indeed that in a manifold of dimension greater than 3 one may, by smooth deformation, reduce the number of self-intersecting points of a minimal horizontal curve. But this is certainly not possible in dimension 2 or 1. In particular, when the basis is a circle there is only one horizontal curve c between two given points, and it is not difficult to find a connection such that c_* self-intersects infinitely many times. This is what motivates the following example.

¶Thanks to R. Montgomery[15] for illuminating discussions on this matter.

V The example $C^\infty(S^1)\otimes M_2(\mathbb{C})$

Let us summarize our comparative analysis of d and d_H. When the holonomy is trivial the two distances are equal by proposition IV.1. When the holonomy is non-trivial we have both:

- a sufficient, but maybe not necessary, condition (Corollary III.4) that guarantees the two distances have the same connected components,

-a necessary condition (Proposition IV.4) for the two distances to coincide on a given connected component.

These two conditions do not seem to be related: writing Q^i and Q^i_H the connected components of d and d_H respectively, it is likely that in some situations $Q^i = Q^i_H$ for some i although d differs from d_H on Q^i, or on the contrary $Q^i_H \subsetneq Q^i$ but $d = d_H$ on Q^i_h. In the present section we exhibit a simple low-dimensional example in which the Q^i's are two dimensional tori (Proposition V.1) and the Q^i_H's are dense subsets. d coincides with d_H only on some part of Q^i_H (Corollary VI.2). The present section is technical and deals with the exact computation of the noncommutative distance (Proposition V.4). Interpretation and discussion are postponed to the following section.

Consider the trivial $U(2)$-bundle P over the circle S^1 of radius one with fiber $\mathbb{C}P^1$, that is to say the set of pure states of $\mathcal{A} = \mathcal{A}_E\otimes\mathcal{A}_I$ with $\mathcal{A}_E = C^\infty(S^1)$ and $\mathcal{A}_I = M_2(\mathbb{C})$, namely

$$\mathcal{A} = C^\infty(S^1, M_2(\mathbb{C})).$$

Let us equip P with a connection whose associated 1-form $A\in\mathfrak{u}(2)$ is constant. For simplicity we restrict to a matrix A of rank one but the adaptation to a wider class of connections should be quite straightforward. Once and for all we fix a basis of $\mathbb{C}^2$ in which the fundamental representation of A is written

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -i\theta \end{pmatrix} \tag{45}$$

where $\theta\in]0,1[$ is a fixed real parameter. Let $[0,2\pi[$ parameterize the circle and call x the point with coordinate 0. Within a trivialization (π, V) the horizontal lift c of the curve

$$c_*(\tau) = \tau \text{ mod } [2\pi], \qquad \tau\in]-\infty,+\infty[\tag{46}$$

with initial condition

$$V(c(0)) = \xi = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{C}P^1$$

is the helix $c(\tau) = (c_*(\tau), V(\tau))$, where

$$V(\tau) = \begin{pmatrix} V_1 \\ V_2 e^{i\theta\tau} \end{pmatrix}. \tag{47}$$

The points of P accessible from $\xi_x = \xi_0 \doteq (\omega_{c_*(0)}, \omega_\xi)$ are the pure states

$$\xi_\tau \doteq (\omega_{c_*(\tau)}, \omega_{V(\tau)}). \tag{48}$$

By the Hopf fibration the fiber $\mathbb{C}P^1$ is seen to be a two sphere. Explicitly ξ is the point of S^2 with Cartesian coordinates

$$x_\xi = 2\mathrm{Re}(V_1\overline{V_2}),\; y_\xi = 2\mathrm{Im}(V_1\overline{V_2}),\; z_\xi = |V_1|^2 - |V_2|^2. \tag{49}$$

Writing

$$2V_1\overline{V_2} \doteq Re^{i\theta_0} \tag{50}$$

one obtains ξ_x as the point in the fiber $\pi^{-1}(x)$ with coordinates

$$x_0 = R\cos\theta_0, \quad y_0 = R\sin\theta_0, \quad z_0 = z_\xi.$$

The points in the fiber over $c_*(\tau)$ that are accessible from ξ_x are

$$\xi^k_\tau \doteq \xi_{\tau+2k\pi}, \quad k \in \mathbb{Z}, \tag{51}$$

with Hopf coordinates

$$x^k_\tau \doteq R\cos(\theta_0 - \theta^k_\tau), \quad y^k_\tau \doteq R\sin(\theta_0 - \theta^k_\tau), \quad z^k_\tau \doteq z_\xi \tag{52}$$

where

$$\theta^k_\tau \doteq \theta(\tau + 2k\pi).$$

All the ξ^k_τ's are on the circle S_R of radius R located at the "altitude" z_ξ in $\pi^{-1}(c_*(\tau))$. Therefore

$$\mathrm{Acc}(\xi_x) \subset \mathbb{T}_\xi$$

where

$$\mathbb{T}_\xi \doteq S^1 \times S_R \tag{53}$$

is the two-dimensional torus (see Figure 2). Similarly for any $\zeta \in \mathbb{C}P^1$ such that $z_\xi = z_\zeta$

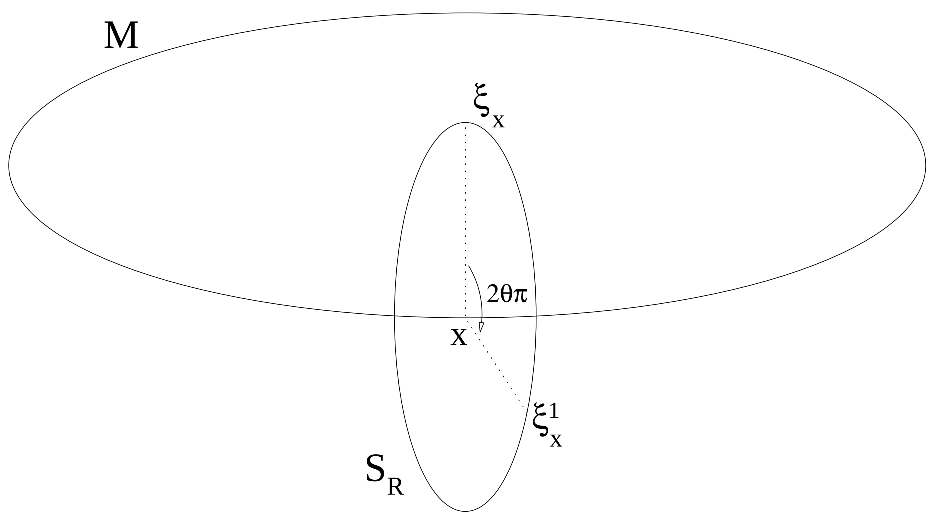

Figure 2: The 2-torus $\mathbb{T}_\xi$. The ξ^k_x's form a dense subset of S_R.

one has $\mathrm{Acc}(\zeta_x) \subset \mathbb{T}_\xi$. In fact

$$\mathbb{T}_\xi = \bigcup_{\substack{\zeta\in\mathbb{C}P^1,\\ z_\zeta = z_\xi}} \mathrm{Acc}(\zeta_x). \tag{54}$$

Note that when θ is irrational $\mathbb{T}_\xi$ is the completion of $\mathrm{Acc}(\xi_x)$ with respect to the Euclidean norm on each S_R.

Proposition V.1 *$\mathbb{T}_\xi$ is the connected component of ξ_x for d.*

Proof. Let $a_{ij} \in \mathcal{A}_E$, $i, j = 1, 2$, be the components of a selfadjoint element of $\mathcal{A}$. (46) yields an explicit identification of $\mathcal{A}_E$ with the algebra of 2π-periodic complex functions on $\mathbb{R}$,

$$a_{ij}(\tau) \doteq a_{ij}(c_*(\tau)) = a_{ij}(\tau + 2k\pi) \quad k \in \mathbb{Z} \tag{55}$$

with

$$a_{ij}(0) = a_{ij}(x).$$

Let dot denote the derivative. Since $M = S^1$ is 1-dimensional, the Clifford action reduces to multiplication by 1 ($\gamma^\mu = \gamma^1 = 1$) and $[D_E, a_{ij}] = -i\dot{a_{ij}}$. Therefore

$$i[\mathcal{D}, a] = \begin{pmatrix} \dot{a_{11}} & \dot{a_{12}} + i\theta a_{12} \\ \dot{a_{21}} - i\theta a_{21} & \dot{a_{22}} \end{pmatrix} \tag{56}$$

is zero if and only if $a_{11} = C$, $a_{22} = C'$ are constant and $a_{12} = \overline{a_{21}} = 0$ ($\dot{a_{12}} = -i\theta a_{12}$ has no other 2π-periodic solution than zero). Under these conditions

$$\xi_x(a) = |V_1|^2 C + (1 - |V_1|^2) C'$$

differs from $\zeta_y(a)$ if and only if $z_\zeta \neq z_\xi$. Hence, identifying a_{ij} with $\lim\limits_{n\to+\infty} (a_n)_{ij}$ in Lemma III.2, one obtains that $d(\xi_x, \zeta_y)$ is infinite if and only if $z_\xi \neq z_\zeta$, that is to say $\zeta_y \notin \mathbb{T}_\xi$. ■

By the proposition above the connected component $\mathbb{T}_\xi$ of d contains, but is distinct from, the connected component $\mathrm{Acc}(\xi_x)$ of d_H. This is enough to establish that the two metrics are not equal. Furthermore the results of the previous section strongly suggest that even on $\mathrm{Acc}(\xi_x)$ the two metrics cannot coincide more than partially. To fix notation let us consider the distance $d(\xi_x, \xi_\tau)$ with $\xi_\tau \in \mathrm{Acc}(\xi_x)$ given by (48) with $\tau > 0$. On the one hand the function on $\mathrm{Acc}(\xi_x)$

$$L(c(\tau)) \doteq d_H(\xi_x, c(\tau)) = \tau \tag{57}$$

is not 2π-periodic, hence not in $\mathcal{A}_E$. Therefore it cannot be used as in (37) to realize the upper bound d_H provided by Proposition III.1. Instead one could be tempted to use the geodesic distance on S^1,

$$F(\tau) \doteq d_{\text{geo}}(\xi_x, c_*(\tau)) = \min(\tau \bmod [2\pi],\ (2\pi - \tau) \bmod [2\pi]), \tag{58}$$

but it may help in proving that $d = d_H$ only as long as as d_H equals d_{geo}, that is to say as long as $\tau \leq \pi$. Similarly $L \bmod [2\pi]$ could be efficient till $\tau = 2\pi$ but it has infinite derivative at $2k\pi$ so it cannot be approximated by some a_n satisfying the commutator norm condition. On the other hand for fixed $k \in \mathbb{Z}$ the projection of the minimal horizontal curve between ξ_τ^k and ξ_τ is a K-fold loop with

$$K = \begin{cases} |k| & \text{for} \quad \theta \text{ irrational} \\ \min\{|k|, ||k| - q|\} & \text{for} \quad \theta = \frac{p}{q} \end{cases}$$

where we assume that p and q are positive relatively prime with respect to each other and kp is not a multiple of q (otherwise ξ_τ^k coincides with ξ_τ). In any case when $|k| = 1$ then $K = 1$ and Proposition IV.4 should not forbid $d(\xi_\tau, \xi_\tau^{\pm 1})$ from equalling $d_H(\xi_\tau, \xi_\tau^{\pm 1}) = 2\pi$.

We show below that this is indeed the case but only when $R = 1$. On the contrary as soon as $K > 3$ Proposition IV.4 certainly forbids d from equalling d_H. In fact the situation is even more restrictive due to the particular choice (45) of the connection. Since the latter commutes with the diagonal part a_1 of any element $a \in \mathcal{A}$, $\xi_\tau^k(a_1) = \xi_\tau(a_1)$ for any $k \leq K$. Proposition IV.4 thus can be written as a system of $K+1$ equations

$$\begin{aligned} (\xi_\tau^k - \xi_\tau)(a_o) &= 2k\pi & (59)\\ \xi_\tau(a_o) &= -\xi_\tau(a_1) & (60) \end{aligned}$$

where $a_0 = a - a_1$. (60) simply defines the diagonal part a_1 and one is finally left with K equations (59) constraining the two real components of a_0. Therefore it is most likely that d does not equal d_H as soon as $K > 2$.

To make these qualitative suggestions more precise, let us study the specific example of a "sea-level" (i.e. $z_\xi = 0$) pure state ξ, assuming

$$|V_1| = |V_2| = \frac{1}{\sqrt{2}}. \tag{61}$$

All the distances on the associated connected component $\mathbb{T}_\xi$ can be explicitly computed. To do so it is convenient to isolate the part of the algebra that really enters the game in the computation of the distances. This is the objective of the following two lemmas. The first one is of algebraic nature: it deals with our explicit choice $\mathcal{A}_I = M_2(\mathbb{C})$ and does not rely on the choice $M = S^1$.

Lemme V.2 *Given ζ_y in $\mathbb{T}_\xi$, the search for the supremum in the computation of $d(\xi_x, \zeta_y)$ can be restricted to the set of elements*

$$a = f\mathbb{I} + a_0 \tag{62}$$

where $\mathbb{I}$ is the identity of $M_2(\mathbb{C})$, $f \in \mathcal{A}_E$ vanishes at x and is positive at y, while a_0 is an element of $\mathcal{A}$ whose diagonal terms are both zero and such that

$$\zeta_y(a_0) - \xi_x(a_0) \geq 0. \tag{63}$$

Proof. Let $\tilde{}$ denote the operation that permutes the elements on the diagonal. By (56)

$$\|[\mathcal{D}, a]\| = \max_{\pm} \left\| \frac{(\dot{a_{11}} + \dot{a_{22}}) \pm \sqrt{(\dot{a_{11}} - \dot{a_{22}})^2 + 4|\dot{a_{12}} + i\theta a_{12}|^2}}{2} \right\| \tag{64}$$

is invariant under the permutation of a_{11} and a_{22}. Thus $\|[\mathcal{D}, a]\| = \|[\mathcal{D}, \tilde{a}]\|$ so

$$\left\| [\mathcal{D}, \frac{\tilde{a} + a}{2}] \right\| \leq \|[\mathcal{D}, a]\|. \tag{65}$$

Meanwhile

$$\xi_x(\frac{\tilde{a} + a}{2}) = \xi_x(a) \quad \text{and} \qquad \zeta_y(\frac{\tilde{a} + a}{2}) = \zeta_y(a) \tag{66}$$

therefore the supremum in the distance formula can be sought on

$$\mathcal{A} + \tilde{\mathcal{A}} = C^\infty(S^1) \otimes \mathbb{I} + \mathcal{A}_0$$

where $\mathcal{A}_0$ is the set of selfadjoint elements of $\mathcal{A}$ whose diagonal terms are zero. This fixes eq.(62). Now if $a = f\mathbb{I} + a_0$ attains the supremum then so does $a - f(x)\mathbb{I}$, hence the vanishing of f at x. Moreover

$$\|[\mathcal{D}, f\otimes\mathbb{I}]\| = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [\mathcal{D}, a] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \quad \leq \quad \|[\mathcal{D}, a]\| \tag{67}$$

$$\|[\mathcal{D}, a_0]\| \leq \left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} [\mathcal{D}, a] \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\| \quad \leq \quad \|[\mathcal{D}, a]\|\,, \tag{68}$$

so when a satisfies the commutator norm condition so do $f\mathbb{I}$ and a_0. This implies that $|\xi_x(a_0) - \zeta_y(a_0)|$ and $|\xi_x(f\mathbb{I}) - \zeta_y(f\mathbb{I})| = |f(y)|$ are smaller than

$$|\xi_x(a) - \zeta_y(a)| = |f(y) + \zeta_y(a_0) - \xi_x(a_0)|. \tag{69}$$

In particular, $f(y)$ and $\zeta_y(a_0) - \xi_x(a_0)$ have the same sign, which we assume positive (if not, consider $-a$ instead of a). ■

Other simplifications come from the choice of S^1 as the base manifold. Especially the following lemma makes clear the role played by the functions L and F discussed in (57,58).

Lemme V.3 *Let $a = f\mathbb{I} + a_0$ satisfy the commutator norm condition, then*

$$\left\|\dot{f}\right\| \leq 1 \quad \textit{and} \quad |f(\tau)| \leq \left\|\dot{f}\right\| F(\tau) \tag{70}$$

where $F(\tau)$ is the 2π-periodic function defined on $[0, 2\pi[$ by

$$F(\tau) \doteq min\,(\tau, 2\pi - \tau)\,. \tag{71}$$

Meanwhile

$$a_0 = \begin{pmatrix} 0 & ge^{-i\theta L} \\ \overline{g}e^{i\theta L} & 0 \end{pmatrix} \tag{72}$$

where $L(\tau) = \tau$ for all τ in $\mathbb{R}$ and g is a smooth function on $\mathbb{R}$ given by

$$g(\tau) = g(0) + \int_0^\tau \rho(u) e^{i\phi(u)} du \tag{73}$$

with $\rho \in C^\infty(\mathbb{R}, \mathbb{R}^+)$, $\|\rho\| \leq 1$, and $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\rho(u + 2\pi) e^{i\phi(u+2\pi)} = \rho(u) e^{i(\phi(u) + 2\theta\pi)} \tag{74}$$

while the integration constant is

$$g(0) = \frac{1}{e^{2i\theta\pi} - 1} \int_0^{2\pi} \rho(u) e^{i\phi(u)} du. \tag{75}$$

Proof. (70) comes from the commutator norm condition (67) together with the 2π-periodicity of f (55), namely

$$f(\tau) = \int_0^\tau \dot{f}(u) du = -\int_\tau^{2\pi} \dot{f}(u) du.$$

The explicit form of a_0 is obtained by noting that any complex smooth function $a_{12} \in \mathcal{A}_E$ can be written $ge^{-i\theta L}$ where $g \doteq a_{12}e^{i\theta L} \in C^\infty(\mathbb{R})$ satisfies

$$g(\tau + 2\pi) = g(\tau)e^{2i\theta\pi}. \tag{76}$$

Hence any selfadjoint a_0 can be written as in (72), which yields for the commutator

$$[\mathcal{D}, a_0] = -i \begin{pmatrix} 0 & \dot{g}e^{-i\theta L} \\ \dot{\bar{g}}e^{i\theta L} & 0 \end{pmatrix}. \tag{77}$$

By (68) the commutator norm condition implies $\|\dot{g}\| \leq 1$, that is to say

$$g(\tau) = g(0) + \int_0^\tau \rho(u)e^{i\phi(u)}du \tag{78}$$

where $\rho \in C^\infty(\mathbb{R}, \mathbb{R}^+)$, $\|\rho\| \leq 1$, $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$. The integration constant is fixed by (76),

$$g(0) = \frac{1}{e^{2i\theta\pi} - 1}\left(\int_0^{\tau+2\pi} \rho(u)e^{i\phi(u)}du - \int_0^\tau \rho(u)e^{i(\phi(u)+2\theta\pi)}du\right), \tag{79}$$

and one extracts (74) from $\frac{d}{d\tau}g(0) = 0$. Reinserted in (79) it finally yields (75). ■

These lemmas yield the main result of the section: the computation of all distances on $\mathbb{T}_\xi$.

Proposition V.4 *Let P be the $\mathbb{C}P^1$ trivial bundle over the circle S^1 of radius one with connection (45). Let ξ_x be a point in P at altitude $z_\xi = 0$ and $\mathbb{T}_\xi$ its connected component for the noncommutative geometry distance d. For any $\zeta_y \in \mathbb{T}_\xi$ there exists an equivalence class of real couples $(\tau, \theta') \sim (\tau + 2\mathbb{Z}\pi, \theta' - 2\mathbb{Z}\theta\pi)$ such that*

$$\zeta_y = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta'} \end{pmatrix} \xi_\tau \tag{80}$$

where ξ_τ is given in (48,47). Without loss of generality one may assume that τ is positive (if not, permute the role played by ξ_x and ζ_y) so that

$$\tau = 2k\pi + \tau_0 \tag{81}$$

with $k \in \mathbb{N}$ and $0 \leq \tau_0 \leq 2\pi$. Then

$$d(\xi_x, \zeta_y) = \begin{cases} \max\,(\,X; X + \tau_0 Y\,) & \text{when } \tau_0 \leq \pi \\ \max\,(\,X; X + (2\pi - \tau_0)Y\,) & \text{when } \pi \leq \tau_0 \end{cases} \tag{82}$$

in which

$$X \doteq RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0) \tag{83}$$

$$Y \doteq 1 - RW_{k+1} - RW_k \tag{84}$$

with R defined in (50) and

$$W_k \doteq \frac{|\sin(k\theta\pi + \frac{\theta'}{2})|}{|\sin\theta\pi|} \tag{85}$$

do not depend on the choice of the representantive of the equivalence class (τ, θ').

Proof. The form (80) of ζ_y comes from the definition (54) of $\mathbb{T}_\xi$. It gives, for an element a of Lemma V.3,

$$|\xi_x(a) - \zeta_y(a)| = f(\tau) + \Re\left(Re^{-i\theta_0}(g(\tau)e^{i\theta'} - g(0))\right) \tag{86}$$

where we use the definition (50) of θ_0, the vanishing of f at x, the positivity of $f(y) = f(\tau)$ as well as (63). The explicit form (73) of g allows us to rewrite (86) as

$$f(\tau) + R\int_0^\tau \rho(u)\cos(\phi'(u)) + \Re\left(Re^{-i\theta_0}g(0)\left(e^{i\theta'} - 1\right)\right) \tag{87}$$

where

$$\phi'(u) \doteq \phi(u) - \theta_0 + \theta'.$$

The point is to find the maximum of (87) on all the 2π-periodic f satisfying (70), the positive ρ, $\|\rho\| \leq 1$ and the ϕ satisfying (74). To do so we will first find an upper bound (eqs. (105) and (106) below) and prove that it is the lowest one.

Fixing a pure state ζ_y means fixing two values θ' and τ or, equivalently by (81), fixing θ', k and τ_0. The integral term in (87) then splits into

$$\Re\int_0^{2k\pi} \rho(u)e^{i\phi'(u)}du = \Re\left(\sum_{n=0}^{k-1} e^{2in\theta\pi}\int_0^{2\pi} \rho(u)e^{i\phi'(u)}du\right) \tag{88}$$

and

$$\Re\int_{2k\pi}^{2k\pi+\tau_0} \rho(u)e^{i(\phi'(u))}du = \Re\left(e^{2ik\theta\pi}\int_0^{\tau_0} \rho(u)e^{i\phi'(u)}du\right) \tag{89}$$

that recombine as

$$S_{k+1}\int_0^{\tau_0} \rho(u)\cos\phi_k(u)\,du + S_k\int_{\tau_0}^{2\pi} \rho(u)\cos\phi_{k_1}(u)\,du \tag{90}$$

where

$$S_k \doteq \frac{\sin k\theta\pi}{\sin\theta\pi} \;\text{ and }\; \phi_k(u) \doteq \phi'(u) + k\theta\pi. \tag{91}$$

To compute the real-part term of (87) one uses the definition (75) of $g(0)$ and obtain

$$S_{\frac{1}{2}}\int_0^{2\pi} \rho(u)\cos\phi_{\frac{1}{2}}(u)\,du \tag{92}$$

where

$$S_{\frac{1}{2}} \doteq \frac{\sin\theta'/2}{\sin\theta\pi} \;\text{ and }\; \phi_{1/2}(u) \doteq \phi'(u) - \frac{\theta'}{2} - \theta\pi.$$

(87) is rewritten as

$$|\xi_x(a) - \zeta_y(a)| = f(\tau) + R\int_0^{\tau_0} \rho(u)G_{k+1}(u)\,du + R\int_{\tau_0}^{2\pi} \rho(u)G_k(u)\,du \tag{93}$$

with

$$G_k \doteq S_k\cos\phi_{k-1} + S_{\frac{1}{2}}\cos\phi_{\frac{1}{2}}. \tag{94}$$

The split of the integral makes the search for the lowest upper bound easier. Calling W_k the maximum of $|G_k(u)|$ on $[0, 2\pi[$, the positivity of ρ makes (93) bounded by

$$f(\tau) + RW_{k+1}\int_0^{\tau_0} \rho(u)\,du + RW_k\int_{\tau_0}^{2\pi} \rho(u)\,du. \tag{95}$$

Now (64) with $a_{11} = a_{22} = f$ and $|\dot{a_{21}} + i\theta a_{12}| = \rho$ yields

$$|\dot{f}(u) + \rho(u)| \leq 1 \quad \text{whenever} \quad |\dot{f}(u)| \geq 0 \tag{96}$$
$$|\dot{f}(u) - \rho(u)| \leq 1 \quad \text{whenever} \quad |\dot{f}(u)| \leq 0 \tag{97}$$

for any $u \in \mathbb{R}$, that is to say

$$\rho \leq 1 - |\dot{f}|. \tag{98}$$

Therefore

$$\int_0^{\tau_0} \rho(u)\, du \leq \tau_0 - \int_0^{\tau_0} |\dot{f}|. \tag{99}$$

Moreover $f(\tau) = f(\tau_0)$ (2π-periodicity of f) is positive by Lemma V.2 so

$$-f(\tau_0) = -|f(\tau_0)| \geq -\int_0^{\tau_0} |\dot{f}(u)|\, du. \tag{100}$$

Hence (99) gives

$$\int_0^{\tau_0} \rho(u)\, du \leq \tau_0 - f(\tau_0). \tag{101}$$

Similarly

$$\int_{\tau_0}^{2\pi} |\dot{f}(u)| du \geq |\int_{\tau_0}^{2\pi} \dot{f}(u) du| = |-\int_0^{\tau_0} \dot{f}(u)\, du| = f(\tau_0)$$

hence

$$\int_{\tau_0}^{2\pi} \rho(u)\, du \leq 2\pi - \tau_0 - f(\tau_0). \tag{102}$$

Back to (95), equations (101) and (102) yield the bound

$$f(\tau_0)Y + X \tag{103}$$

where X is defined in (83) and Y in (84). By (70) and in case

$$Y \geq 0, \tag{104}$$

(103) yields

$$|\xi_x(a) - \zeta_y(a)| \leq \begin{cases} X + \tau_0 Y & \text{for } 0 \leq \tau_0 \leq \pi \\ X + (2\pi - \tau_0)Y & \text{for } \pi \leq \tau_0 \leq 2\pi \end{cases}. \tag{105}$$

When $Y \leq 0$,

$$|\xi_x(a) - \zeta_y(a)| \leq X. \tag{106}$$

These are the announced lowest upper bounds. To convince ourselves let us build a sequence a_n that realizes (105) or (106) at the limit $n \to +\infty$. As a preliminary step note that an easy calculation from (94) yields

$$G_k = A_k \cos\phi' + B_k \sin\phi'$$

where

$$A_k \doteq S_{\frac{1}{2}} \cos(\frac{\theta'}{2} + \theta\pi) + S_k \cos(k-1)\theta\pi \tag{107}$$
$$B_k \doteq S_{\frac{1}{2}} \sin(\frac{\theta'}{2} + \theta\pi) - S_k \sin(k-1)\theta\pi. \tag{108}$$

G_k attains its maximum value

$$W_k \doteq |A_k|\sqrt{1+\frac{|B_k|^2}{|A_k|^2}} = \frac{|\sin(k\theta\pi+\frac{\theta'}{2})|}{|\sin(\theta\pi)|} \tag{109}$$

when[‖]

$$\phi' = \Phi_k \doteq \mathrm{Arctan}\frac{B_k}{A_k} \;\text{ or }\; \mathrm{Arctan}\frac{B_k}{A_k}+\pi. \tag{110}$$

Let then

$$a_n = \begin{pmatrix} f_n & g_n e^{-i\theta L} \\ \overline{g}_n e^{i\theta L} & f_n \end{pmatrix}$$

be a sequence of elements of $\mathcal{A}$ that depend on the fixed value $\tau = 2k\pi+\tau_0$ in the following way: in case (104) is fulfilled and $\tau_0 \le \pi$, f_n approximates from below the 2π-periodic function

$$f_-(t) = \begin{cases} t & \text{for } 0 \le t \le \tau_0 \\ \tau_0 - C(t-\tau_0) & \text{for } \tau_0 \le t \le 2\pi \end{cases} \tag{111}$$

with

$$C \doteq \frac{\tau_0}{2\pi-\tau_0}.$$

In case $\tau_0 \ge \pi$, f_n approximates

$$f_+(t) = \begin{cases} \frac{t}{C} & \text{for } 0 \le t \le \tau_0 \\ 2\pi - t & \text{for } \tau_0 \le t \le 2\pi \end{cases}. \tag{112}$$

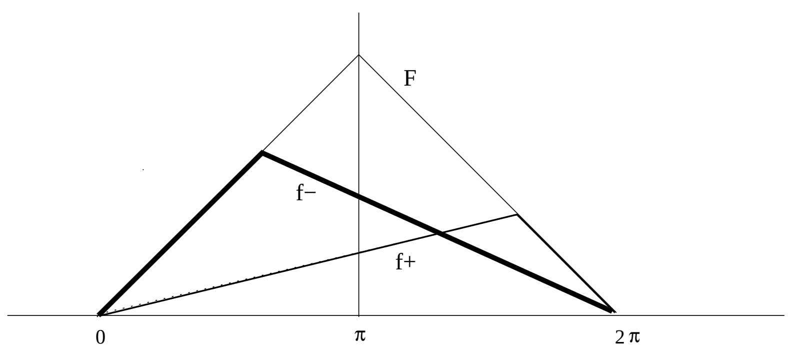

Figure 3: The functions F, f_+ and f_-.

When (104) is not fulfilled, $f_n = f_0$ is simply the zero function. In any case and whatever τ_0, g_n is defined via (73) and (75), replacing ϕ with a sequence ϕ_n approximating the step function Φ of width 2π and height $2\theta\pi$ defined on $[0, 2\pi[$ by

$$\Phi(u) = \begin{cases} \Phi_{k+1} + \theta_0 - \theta' & \text{for } 0 \le u < \tau_0 \\ \Phi_k + \theta_0 - \theta' & \text{for } \tau_0 < u < \pi \end{cases}, \tag{113}$$

[‖] The ambiguity in the explicit form of Φ_k is not relevant. Depending on the respective signs of A_k and B_k, one choice yields W_k whereas the other one yields $-W_K$. What is important is the existence of a well defined value Φ_k such that $A_k \cos\Phi_k + B_k \sin\Phi_k = W_k$.

and replacing ρ with a sequence ρ_n approximating the 2π-periodic function

$$\Gamma_I = 1 - |\dot{f}_I| \tag{114}$$

where $I = +, -$ or 0. By construction the a_n's satisfy the commutator norm condition. In particular the fact that $\lim\limits_{n\to+\infty} \rho_n$ and Φ are step functions is not problematic since their derivatives are not constrained by the commutator. For technical details on how to approximate step functions by sequences of smooth functions, the reader is invited to consult classical textbooks such as [5]. The last point is to check that

$$\lim_{n\to+\infty} |\xi_x(a_n) - \zeta_y(a_n)| = (105). \tag{115}$$

This is a simple notation exercise: (91) gives

$$\begin{aligned} \phi'(u) &= \Phi_{k+1} \quad \text{for } 0 \leq u < \tau_0 && (116)\\ \phi'(u) &= \Phi_k \quad\ \ \text{for } \tau_0 < u < 2\pi. && (117) \end{aligned}$$

Therefore, by (93) together with (114),

$$\begin{aligned} \lim_{n\to+\infty} |\xi_x(a_n) - \zeta_y(a_n)| &= f_I(\tau_0) + RW_{k+1} \int_0^{\tau_0} \Gamma_I(u)\, du + RW_k \int_{\tau_0}^{2\pi} \Gamma_I(u)\, du \\ &= f_I(\tau_0) - RW_{k+1} \int_0^{\tau_0} |\dot{f}_\pm|\, du - RW_k \int_{\tau_0}^{2\pi} |\dot{f}_I|\, du \\ &\quad + RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0). \end{aligned} \tag{118}$$

When (104) is fulfilled and $\tau_0 \leq \pi$, the subscript of f is minus and (111) makes (118) equal to

$$\tau_0 - RW_{k+1}\tau_0 - RW_k(2\pi - \tau_0)C + RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0)$$

which is exactly the first line of (105). Similarly for $\tau_0 \geq \pi$, the subscript turns to + and (112) yields for (118)

$$(2\pi - \tau_0) - RW_{k+1}\frac{\tau_0}{C} - RW_k(2\pi - \tau_0) + RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0),$$

which is nothing but the second line of (105). Finally, when (104) is not fulfilled, $f_I = \dot{f}_I = 0$ and (118) equals (106). ■

Let us check the coherence of our result by noticing that for $\tau_0 = \pi$ both formulas of (82) agree and yield

Check 1 $d(\xi_x, \zeta_\pi) = \max(X, X + \pi Y) = \max(\pi R(W_{k+1} + W_k);\ \pi)$.

Similarly for a given k and $\tau_0 = 2\pi$, the second line of (82) agrees with the first line with $k+1$ and $\tau_0 = 0$, namely

Check 2 $d(\xi_x, \zeta_{2k\pi+2\pi}) = 2\pi RW_{k+1} = d(\xi_x, \zeta_{2(k+1)\pi+0})$.

This is nothing but the restriction of d to the fiber over x. Its extreme simplicity (no "max" is involved) indicates that the noncommutative metric is better understood fiberwise. We shall see in the next section that this is the main difference from the Carnot-Carathéodory

metric. Another check, and certainly the best guarantee that Proposition V.4 is true, is to directly verify that formula (82) does define a metric: the vanishing of d when $\zeta_y = \xi_x$ is obvious; the invariance under the exchange $\xi_x \longleftrightarrow \zeta_y$ is not testable since the symmetry $\tau \longleftrightarrow -\tau$ is broken from the beginning by the specification that τ is positive. There remains the triangle inequality.

Check 3 *For any* $\zeta_1, \zeta_2 \in \mathbb{T}_\xi$, $d(\xi_x, \zeta_2) \leq d(\xi_x, \zeta_1) + d(\zeta_1, \zeta_2)$.

Proof. Let ζ_{τ_i}, $i = 1, 2$, be two pure states defined by $\tau_i = 2\pi k_i + t_i$ and θ'_i, labeled in such a way that $\tau_1 \leq \tau_2$. The point is to check that

$$\Delta \doteq d(\xi_x, \zeta_{\tau_1}) + d(\zeta_{\tau_1}, \zeta_{\tau_2}) - d(\xi_x, \zeta_{\tau_2}) \tag{119}$$

is positive. Proposition V.4 is invariant under translation (i.e. a reparameterization of the circle $\tau \to \tau + \text{constant}$), which means that $d(\zeta_{\tau_1}, \zeta_{\tau_2})$ is given by formula (82) with W_k replaced by

$$W_{k_{12}} \doteq \frac{|\sin(k_{12}\theta\pi + \frac{\theta'_2 - \theta'_1}{2})|}{|\sin\theta\pi|}$$

and τ_0 replaced by t_{12}. Here k_{12} and t_{12} are such that $\tau_{12} \doteq \tau_2 - \tau_1 \doteq 2k_{12}\pi + t_{12}$. Explicitly

$$k_{12} = k_2 - k_1, \qquad t_{12} = t_2 - t_1 \qquad \text{if } t_1 \leq t_2 \tag{120}$$

$$k_{12} = k_2 - k_1 - 1, \quad t_{12} = 2\pi + t_2 - t_1 \quad \text{if } t_2 \leq t_1. \tag{121}$$

Let X_i, Y_i, $i \in \{1, 2, 12\}$, denote (83) and (84) in which k is replaced by k_i. The only difficulty in checking that (119) is positive is the quite large number of possible expressions for Δ: one for each combination of the signs of the Y_i's and $t_i - \pi$. A simple way to reduce the number of cases under investigation is to decorate Δ with three arrows indicating whether Y_1, Y_{12} and Y_2 respectively are positive (upper arrow) or negative (lower arrow). For instance $\Delta_{\uparrow\uparrow\downarrow}$ denotes the value of Δ when $Y_1 \geq 0$, $Y_{12} \geq 0$, and $Y_2 \leq 0$. Let us also use $\tilde{\Delta}$ decorated with arrows to denote the formal expression (119) in which $d(\xi_x, \xi_{\tau_1})$, $d(\zeta_{\tau_1}, \zeta_{\tau_2})$ and $d(\xi_x, \zeta_{\tau_2})$ are replaced either by $X_i + t_i^m Y_i$ (upper arrow) or by X_i (lower arrow). Here $t_i^m \doteq \min(t_i, 2\pi - t_i)$. For instance

$$\tilde{\Delta}_{\uparrow\uparrow\uparrow} = \tilde{\Delta}_{\downarrow\uparrow\uparrow} + t_1^m Y_1 \tag{122}$$

$$= \tilde{\Delta}_{\uparrow\downarrow\uparrow} + t_2^m Y_2 \tag{123}$$

$$= \tilde{\Delta}_{\downarrow\downarrow\uparrow} + t_1^m Y_1 + t_2^m Y_2. \tag{124}$$

Now suppose that Y_1, Y_{12}, Y_2 are all positive, then

$$\Delta = \Delta_{\uparrow\uparrow\uparrow} = \tilde{\Delta}_{\uparrow\uparrow\uparrow} \geq \begin{cases} \tilde{\Delta}_{\downarrow\uparrow\uparrow} \\ \tilde{\Delta}_{\uparrow\downarrow\uparrow} \\ \tilde{\Delta}_{\downarrow\downarrow\uparrow} \end{cases}.$$

Changing the sign of $Y_1 \leq 0$ and Y_{12} yields

$$\Delta = \Delta_{\downarrow\downarrow\uparrow} = \tilde{\Delta}_{\downarrow\downarrow\uparrow} \geq \begin{cases} \tilde{\Delta}_{\uparrow\downarrow\uparrow} \\ \tilde{\Delta}_{\downarrow\uparrow\uparrow} \\ \tilde{\Delta}_{\uparrow\uparrow\uparrow} \end{cases}.$$

Therefore, if one is able to show *without using the sign of* Y_1 *or the sign of* Y_{12} that $\tilde{\Delta}_{\uparrow\uparrow\uparrow}$ is positive, one proves that both $\Delta_{\uparrow\uparrow\uparrow}$ and $\Delta_{\downarrow\downarrow\uparrow}$ are positive. In fact showing that one of the $\tilde{\Delta}_{\updownarrow\updownarrow\uparrow}$'s is positive is enough to prove that all the $\Delta_{\updownarrow\updownarrow\uparrow}$'s are positive (here $\updownarrow$ means either $\uparrow$ or $\downarrow$). Of course the same is true with $\tilde{\Delta}_{\updownarrow\updownarrow\downarrow}$ so that, at the end, one just has to check the inequality of the triangle for one of the $\tilde{\Delta}_{\updownarrow\updownarrow\uparrow}$ and one of the $\tilde{\Delta}_{\updownarrow\updownarrow\downarrow}$.

Let us begin by $\tilde{\Delta}_{\updownarrow\updownarrow\downarrow}$, assuming first $t_1 \leq t_2$. With $W_i \doteq W_{k_i}$, $W_{i+1} \doteq W_{k_i+1}$, (120) yields

$$\begin{aligned} R^{-1}\tilde{\Delta}_{\downarrow\downarrow\downarrow} &= W_{1+1}t_1 + W_1(2\pi - t_1) + W_{12+1}t_{12} + W_{12}(2\pi - t_{12}) \\ &\quad - W_{2+1}t_2 - W_2(2\pi - t_2) \\ &= (2\pi - t_2)(W_1 + W_{12} - W_2) + t_{12}(W_1 + W_{12+1} - W_{2+1}) \\ &\quad + t_1(W_{1+1} + W_{12} - W_{2+1}) \end{aligned}$$

which is positive since [**]

$$W_{k_2} \leq W_{k_1} + W_{k_{12}} \tag{125}$$

and similar equations for the other indices. Assuming now $t_2 \leq t_1$, (121) yields

$$\begin{aligned} R^{-1}\tilde{\Delta}_{\downarrow\downarrow\downarrow} &= t_2(W_{1+1} + W_{12+1} - W_{2+1}) + (2\pi - t_1)(W_1 + W_{12+1} - W_2) \\ &\quad + (2\pi - t_{12})(W_{12} + W_{1+1} - W_2) \end{aligned}$$

which is also positive by equations similar to (125) (be careful to use the definition (121) of k_{12} and no longer definition (120)). Thus, whatever t_1 and t_2, $\tilde{\Delta}_{\updownarrow\updownarrow\downarrow}$ is positive and the triangle inequality is checked for all the configurations $\updownarrow\updownarrow\downarrow$ of the Y_i's.

Things are slightly more complicated for the configurations $\updownarrow\updownarrow\uparrow$ for one also has to deal with the signs of $t_i - \pi$. First assume $t_1 \leq t_2$:

- $t_1 \leq t_2 \leq \pi$ (implies $t_{12} \leq \pi$),

$$\begin{aligned} (2R)^{-1}\tilde{\Delta}_{\uparrow\uparrow\uparrow} &= W_1(\pi - t_1) + W_{12}(\pi - t_{12}) - W_2(\pi - t_2) \\ &\geq (\pi - t_2)(W_1 + W_{12} - W_2). \end{aligned}$$

- $\pi \leq t_1 \leq t_2$ (implies $t_{12} \leq \pi$),

$$\begin{aligned} \tilde{\Delta}_{\uparrow\uparrow\uparrow} &= 2R(W_{1+1}(t_1 - \pi) + W_{12}(\pi - t_{12}) - W_{2+1}(t_2 - \pi)) + 2(t_2 - t_1) \\ &\geq 2R(t_1 - \pi)(W_{1+1} + W_{12} - W_{2+1}) + 2(t_2 - t_1)(1 - RW_{2+1}). \end{aligned}$$

- $t_1 \leq \pi \leq t_2$ and $t_{12} \leq \pi$,

$$\tilde{\Delta}_{\uparrow\uparrow\uparrow} = 2R(W_1(\pi - t_1) + W_{12}(\pi - t_{12})) + 2(t_2 - \pi)(1 - RW_{2+1}).$$

- $t_1 \leq \pi \leq t_2$ and $t_{12} \geq \pi$,

$$\begin{aligned} \tilde{\Delta}_{\uparrow\uparrow\uparrow} &= 2R(W_1(\pi - t_1) + W_{12+1}(t_{12} - \pi) - W_{2+1}(t_{12} - \pi)) + t_1(1 - 2RW_{2+1}) \\ &\geq 2R(t_{12} - \pi)(W_1 + W_{12+1} - W_{2+1}) + 2t_1(1 - RW_{2+1}). \end{aligned}$$

These five expressions are positive by (125) and the positivity of Y_2. Similarly, in case $t_2 \leq t_1$:

[**] this comes from $|\sin(a+b)| \leq |\sin a| + |\sin b|$ with $a = (k_2 - k_1)\theta\pi + \theta_2' - \theta_1'$ and $b = k_1\theta\pi + \theta_1'$

- $t_2 \leq t_1 \leq \pi$ (implies $t_{12} \geq \pi$),

$$
\begin{aligned}
\tilde{\Delta}_{\uparrow\uparrow\uparrow} &= 2R(W_1(\pi - t_1) + W_{12+1}(t_{12} - \pi) - W_2(\pi - t_1)) + 2(t_1 - t_2)(1 - RW_2) \\
&\geq 2R(\pi - t_1)\,(W_1 + W_{12+1} - W_2) + 2(t_1 - t_2)\,(1 - RW_2)\,.
\end{aligned}
$$

- $\pi \leq\ t_2 \leq t_1$ (implies $t_{12} \geq \pi$),

$$
\begin{aligned}
(2R)^{-1}\tilde{\Delta}_{\uparrow\uparrow\uparrow} &= W_{1+1}(t_1 - \pi) + W_{12+1}(t_{12} - \pi) - W_{2+1}(t_2 - \pi) \\
&\geq (t_2 - \pi)(W_{1+1} + W_{12+1} - W_{2+1}).
\end{aligned}
$$

- $t_2 \leq \pi \leq t_1$ and $t_{12} \leq \pi$,

$$
\begin{aligned}
\tilde{\Delta}_{\uparrow\uparrow\uparrow} &= 2R(W_{1+1}(t_1 - \pi) + W_{12}(\pi - t_{12}) - W_2(\pi - t_{12})) + 2(2\pi - t_1)(1 - RW_2) \\
&\geq 2R(\pi - t_{12})(W_{1+1} + W_{12} - W_2) + 2(2\pi - t_1)(1 - RW_2).
\end{aligned}
$$

- $t_2 \leq \pi \leq t_1$ and $t_{12} \geq \pi$,

$$
\tilde{\Delta}_{\uparrow\uparrow\uparrow} = 2RW_{1+1}(t_1 - \pi) + 2RW_{12+1}(t_{12} - \pi) + 2(\pi - t_2)(1 - RW_2).
$$

■

The proof above is long but we believe it is important to convince oneself that formula V.4 does define a metric, which is not obvious at first sight. As a final test, let us come back to the beginning of this section and verify Lemma III.1.

Check 4 *$d(\xi_x, \zeta_y) \leq d_H(\xi_x, \zeta_y)$ for any $\zeta_y \in Acc(\xi_x)$.*

Proof. Let $\zeta_y = \xi_\tau$. Then $d_H(\xi_x, \xi_\tau) = 2k\pi + \tau_0$ so that

$$
d(\xi_x, \xi_\tau) - d_H(\xi_x, \xi_\tau) \begin{cases}
= 2RW_k(\pi - \tau_0) - 2k\pi & \\
\quad \leq 2\pi(W_k - k) & \text{when } Y \geq 0,\ \tau_0 \leq \pi, \\
= 2RW_{k+1}(\tau_0 - \pi) - 2(\tau_0 - \pi) - 2k\pi & \\
\quad \leq -2(\tau_0 - \pi)W_k - 2k\pi & \text{when } Y \geq 0,\ \tau_0 \geq \pi, \\
= \tau_0(RW_{k+1} - RW_k - 1) & \\
\quad +2\pi(RW_k - k) & \text{when } Y \leq 0.
\end{cases}
\tag{126}
$$

These three expressions are negative by (125) and †† $|\sin k\theta\pi| \leq k|\sin\theta\pi|$. ■

VI Interpretation: a smooth cardio-torus

This section aims at analyzing the result of Proposition V.4. We first compare d to d_H on $\mathrm{Acc}(\xi_x)$ (corollaries VI.1 and VI.2), then study the restriction of d to the fiber over x and to the base $M = S^1$. The reader may wonder why we do not systematically replace R by its value 1. The point is that for two states on the same fiber ($y = x$) the diagonal part of a does not play any role so that Proposition V.4 is valid also for non vanishing z_ξ. Also, for $y \neq x$ some calculations[13] show that V.4 is still valid for non-zero z_ξ as long as $2V_i^2 - R(W_{k+1} + W_k)$ is positive for both $i = 1, 2$. This is the reason why, in the following discussion, we keep writing R.

†† obvious for $k \leq 1$, then by induction

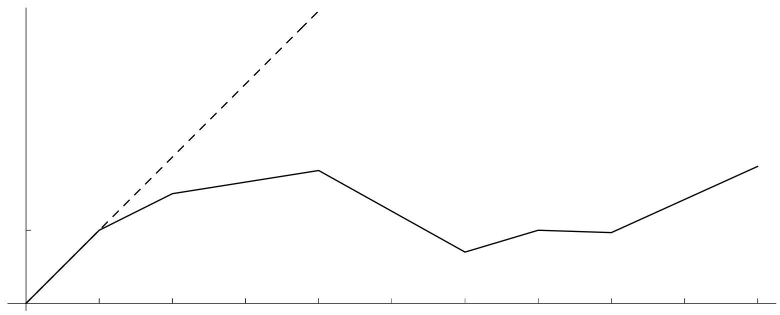

Figure 4: $d(\xi_x, \xi_\tau)$ as a function of τ for $\theta = \frac{1}{\sqrt{2}}$, $R = 0.75$. The dashed line is $d_H(\xi_x, \xi_\tau)$. The unit is π.

VI.1 The shape of $\mathbb{T}_\xi$

Taking ζ_y in $\mathrm{Acc}(\xi_x)$ amounts to setting $\theta' = 0$. W_k is replaced by

$$S_k \doteq \frac{|\sin k\theta\pi|}{|\sin \theta\pi|}$$

and proposition V.4 is rewritten in a somehow more readable fashion.

Corollary VI.1 *Let $\zeta_y = \xi_\tau \in Acc(\xi_x)$, with $\tau = 2k\pi + \tau_0$. For k such that $S_{k+1} + S_k \leq \frac{1}{R}$,*

$$d(\xi_x, \xi_\tau) = \begin{cases} 2RS_k(\pi - \tau_0) + \tau_0 & \textit{when } \tau_0 \leq \pi \\ 2RS_{k+1}(\tau_0 - \pi) + 2\pi - \tau_0 & \textit{when } \pi \leq \tau_0 \end{cases}. \tag{127}$$

For k such that $S_{k+1} + S_k \geq \frac{1}{R}$,

$$d(\xi_x, \xi_\tau) = RS_{k+1}\tau_0 + RS_k(2\pi - \tau_0).$$

It is easy to see on which part of P the noncommutative geometry metric and the Carnot-Carathéodory one coincide.

Corollary VI.2 *For any R, $d(\xi_x, \xi_\tau) = d_H(\xi_x, \xi_\tau)$ for $\tau \in [0, \pi]$. Moreover if $R = 1$ the two metrics are also equal for $\tau \in [\pi, 2\pi]$. These are the only situations in which $d = d_H$.*

Proof. $S_0 = 0$, $S_1 = 1$ and by construction $R \leq 1$. Therefore for $k = 0$, $S_{k+1} + S_k = 1 \leq \frac{1}{R}$ so

$$d(\xi_x, \xi_\tau) = \begin{cases} \tau_0 = d_H(\xi_x, \xi_{\tau_0}) & \text{when } \tau_0 \leq \pi \\ 2\pi(1 - R) + \tau_0(2R - 1) & \text{when } \pi \leq \tau_0 \end{cases} \tag{128}$$

which yields the equality of d and d_H for the indicated values of τ and R. From check 4 in the preceding section, d may equal d_H only if $S_k = k$, i.e. $k = 0$ or 1. When $k = 1$, $S_k + S_{k+1} \geq 1$ and the last line of (126) gives the difference δ between d and d_H,

$$\delta = \tau_0(RS_2 - R - 1) + 2\pi(R - 1). \tag{129}$$

$S_2 \leq 2$ so $\delta \leq (R - 1)(2\pi + \tau)$. δ may vanish only if $R = 1$ and, going back to (129), only if $\tau_0 = 0$. ■

This result is more restrictive that what was expected from Proposition IV.4 revisited in (59), namely that d may equal d_H as long as c does not have sequences of more than 2 self-intersecting points, i.e. up to $\tau_{\max} = 4\pi + \tau_0$. It seems that Proposition IV.4 alone is not sufficient to show that $\tau_{\max} \leq 2\pi$. At best one can obtain

$$\tau_{\max} < 4\pi. \tag{130}$$

Although (130) is not in se an interesting result but simply a weaker formulation of Corollary VI.2, we believe it is interesting to see how far Proposition IV.4 can lead. This could be the starting point for a generalization of the results of this paper to manifolds other than S^1. Let $G, \overline{G}$ be the off-diagonal components of a. (59) is rewritten as

$$\Im\left(G(\tau)e^{i(\theta\tau-\theta_0)}e^{ik\theta\pi}\right) = -\frac{k\pi}{R\sin k\theta\pi} \tag{131}$$

for any $k = 1, ..., K$. For $K = 2$ this system has a unique solution

$$G(\tau) = Ce^{-i\theta\tau}e^{i(\theta_0 - \frac{\pi}{2})} \tag{132}$$

where

$$C \doteq -\frac{2\pi}{R\sin 2\theta\pi}$$

is a constant. Therefore $\xi_\tau(a_0) = \Re\left(e^{i(\theta\tau-\theta_0)}G(\tau)\right) = 0$ so that, by (60), $\xi_\tau(a) = 0$. By Proposition IV.4 this is possible only for $\tau = 0$. Hence there cannot be more than one sequence of 2 self-intersecting points, hence (130).

In any case, when τ is greater than 2π, d strongly differs from d_H. While the latter is unbounded, the former is bounded,

$$d(\xi_x, \zeta_y) \leq \max\left(\frac{2\pi R}{|\sin\theta\pi|}, \pi\right).$$

As illustrated in figure 4, $\mathrm{Acc}(\xi_x)$ viewed as a 1-dimensional object looks like a straight line when it is equipped with d_H, whereas it looks rather chaotic when it is equipped with d.

VI.2 The shape of the fiber

From a fiberwise point of view the situation drastically changes. Parameterizing the fiber S_x over x by

$$\phi \doteq 2k\theta\pi + \theta' \mod [2\pi],$$

one obtains a very simple expression for the noncommutative distance,

$$d(0,\phi) = \frac{2\pi R}{|\sin\theta\pi|}\sin\frac{\phi}{2}. \tag{133}$$

For those points of S_x which are accessible from ξ_x, namely for $\theta' = 0$, the Carnot-Carathéodory metric is

$$d_H(0,\phi) = 2k\pi.$$

Hence, when θ is irrational and in any neighborhood of $\xi_x = 0$ in the Euclidean topology of S_x, it is always possible to find some

$$\phi_k \doteq \xi_0^k = 2k\theta\pi \mod [2\pi]$$

which are arbitrarily Carnot-Carathéodory-far from ξ_x. In other terms d_H destroys the S^1 structure of the fiber. On the contrary d keeps it in mind in a rather intriguing way. Let us compare d to the Euclidean distance d_E on the circle of radius

$$\mathcal{R} \doteq \frac{2R}{|\sin\theta\pi|}. \tag{134}$$

At the cut-locus $\phi = \pi$, the two distances are equal but whereas $d_E(0,.)$ is not smooth, the noncommutative geometry distance *is* smooth (cf Figure 5).

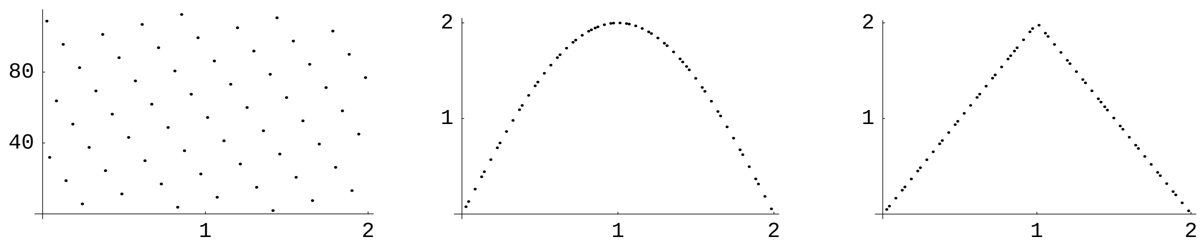

Figure 5: $d_H(0,\phi_k)$, $d(0,\phi_k)$, $d_E(0,\phi_k)$. Vertical unit is $\frac{\pi R}{\sin\theta\pi}$, horizontal unit is π.

In this sense, if we imagine an observer localized at ξ_x and whose only information about the geometry of the surrounding world is the measurement of the function $d(0,\phi)$, S_x looks "smoother than a circle". More rigourously, (133) turns out to be the length $L(\phi)$ of the minimal arc joining the origin to a point ϕ on the cardioid with polar equation

$$r = \frac{\pi\mathcal{R}}{4}(1+\cos\varphi). \tag{135}$$

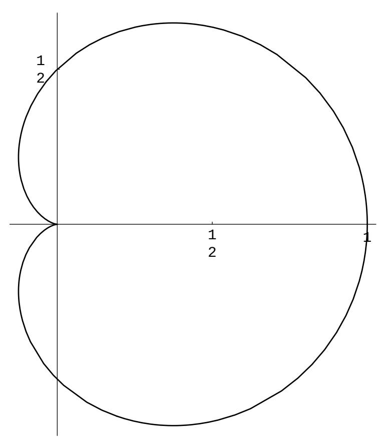

Figure 6: Cardioid $r = \frac{\pi\mathcal{R}}{4}(1+\cos\varphi)$. Units are in $\frac{\pi R}{|\sin\theta\pi|}$.

Indeed restricting to $0 \leq \phi \leq \pi$ (since $L(\phi) = L(2\pi - \phi)$),

$$L(\phi) = \int_0^\phi \sqrt{r^2 + (\frac{dr}{d\varphi})^2}d\varphi = \int_0^\phi \frac{\pi\mathcal{R}}{2}\cos\frac{\varphi}{2}d\varphi = \pi\mathcal{R}\sin\frac{\phi}{2} = d(0,\phi). \tag{136}$$

One has to be careful with the interpretation of equation (136). The noncommutative geometry distance does *not* turn the loop S_x into a cardioid. What the noncommutative metric does is to turn S_x into an object that looks like a cardioid for an observer localized at x who is measuring the distance between him and a point of S_x. Corollary VI.1 being invariant under a re-parameterization of the basis S^1 ($\tau \to \tau + \text{const.}$), the same analysis is true for an observer localized at $y \neq x$. In this sense the cardioid point of view is an intrinsic point of view.

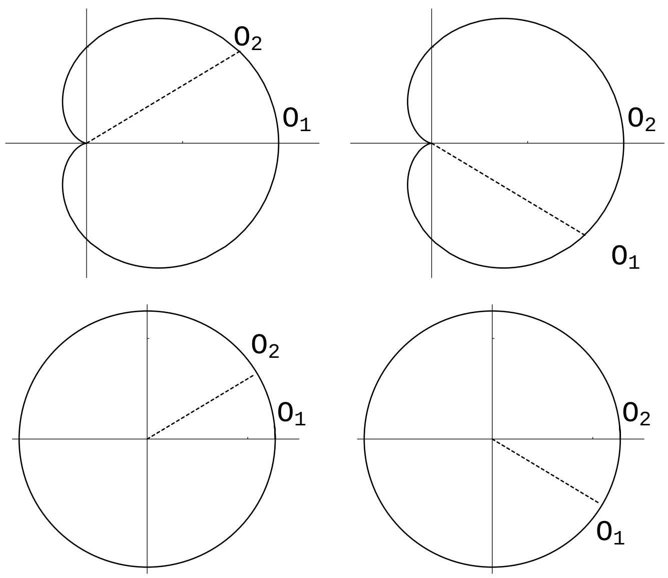

Figure 7: On the left, the loop S according to $\mathcal{O}_1$; on the right, the loop S according to $\mathcal{O}_2$. At bottom S is a circle and one goes from left to right by re-parameterization. On top S is the fiber S_x and a single manifold cannot encompass both points of view.

Things are clearer in analogy with the circle (Figure 7): consider 2 observers $\mathcal{O}_i$, $i = 1, 2$, located at distinct points ϕ_i on a loop S. Assume each of them measures its own distance function

$$d_i : z \in S \mapsto d_i(x_i, z).$$

If both find that $d_i = d_E$, then they will agree that S is a circle. On the contrary if both find that $d_i = d$, then each of them will pretend to be localized at the point opposite to the cut locus of the cardioid and they will disagree on the nature of S. In fact their disagreement is only due to their belief that S is a manifold. What the present work shows is precisely that the loop S_x equipped with the noncommutative metric d is *not* a manifold. This example nicely illustrates how the distance formula (3) allows one to define on very simple objects (like tori) a metric which is not accessible from classical differential geometry.

VI.3 The shape of the basis

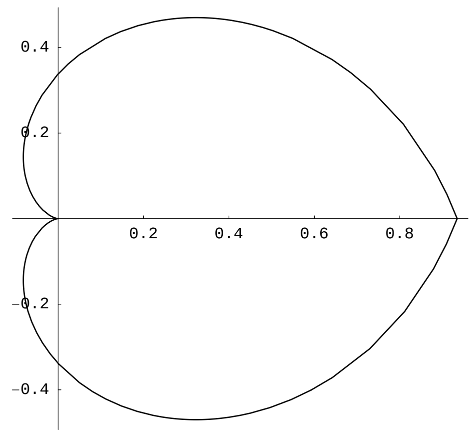

Figure 8: The shape of S_ξ when $\theta \to 1$.

From an intrinsic point of view the fiber looks like a cardioid. What does the base $M = S^1$ look like ? Let S_ξ denote the set of points of $\mathbb{T}_\xi$ corresponding to the same vector $\xi \in \mathbb{C}P^{n-1}$,

$$S_\xi \doteq \{p \in P,\ V(p) = \xi\}.$$

We parameterize S_ξ by $\varphi \in [0, 2\pi[$ with $\xi_x = 0$. Any point in S_ξ can be obtained as a ζ_τ where $\tau = 2k\pi + \varphi$ and ζ defined by (80) with

$$\theta' = -\theta\tau. \tag{137}$$

In order to compute d_H, note that ζ_τ is accessible from ξ_x if and only if ζ_0 is accessible, that is to say iff $\theta' = 2k'\theta\pi \mod [2\pi]$ for some integer k'. In other words $\mathrm{Acc}(\xi_x) \cap S_\xi$ is the subset of $[0, 2\pi[$ given by the numbers φ that write

$$\varphi = 2p\theta^{-1}\pi + 2p'\pi$$

for some integers p, p'. When θ is irrational $\mathrm{Acc}(\xi_x) \cap S_\xi$ is dense in S_ξ and to a given φ corresponds one and only one couple of integers p, p'. By (137) one obtains

$$\zeta_0 = \xi_0^{-(k+p')},$$

where we used the notation (51). Hence $\zeta_\tau = \xi_{2p\theta^{-1}\pi}$, so that

$$d_H(0, \varphi) \doteq d_H(\xi_x, \zeta_\tau) = 2p\theta^{-1}\pi.$$

As in the case of the fiber S_x, one finds close to $0 \in S_\xi$ in the Euclidean topology some points that are infinitely Carnot-Carathéodory far from 0. Hence d_H not only forgets the shape of the fiber but also the shape of the base.

On the contrary the noncommutative distance d is finite on S_ξ and preserves the shape of the base, although the latter is deformed in a slightly more complicated way than the fiber. Note that, via (137),

$$W_k = \frac{|\sin(\frac{\theta}{2}\varphi)|}{|\sin\theta\pi|},\ \ W_{k+1} = \frac{|\sin(\frac{\theta}{2}(2\pi - \varphi))|}{|\sin\theta\pi|}$$

are independent of k. The same is true for X and Y so that $d(0,\phi) = d(\xi_x, \zeta_\tau)$ only depends on φ as expected. Explicitly, defining $\lambda \doteq \frac{\varphi}{2\pi}$, Proposition V.4 writes

$$d(0,\varphi) = \pi\mathcal{R}\left(\lambda \sin(\theta\pi(1-\lambda)) + (1-\lambda)\sin(\theta\pi\lambda)\right) \tag{138}$$

when Y is negative and

$$d(0,\varphi) = \begin{cases} 2\pi\left(\mathcal{R}(\frac{1}{2}-\lambda)\sin\theta\pi\lambda + \lambda\right) & \text{when } \lambda \leq \frac{1}{2} \\ 2\pi\left(\mathcal{R}(\lambda-\frac{1}{2})\sin\theta\pi(1-\lambda) + 1-\lambda\right) & \text{when } \lambda \geq \frac{1}{2}. \end{cases} \tag{139}$$

when Y is positive. Even for a fixed value of R, Y may change sign when φ runs from 0 to 2π so it seems difficult to find for S_ξ a picture like the cardioid for S_x. However, assuming that Y is always negative, one can view the first line of (138) as a kind of convex deformation of a cardioid. In particular when $\theta \to 1$ or $\theta \to 0$, Y is indeed negative for any φ so that

$$\lim_{\theta\to 1} d(0,\varphi) = \pi\mathcal{R}\sin\frac{\varphi}{2}$$

which corresponds to the length on a cardioid of infinite radius (since $\lim\limits_{\theta\to 1}\mathcal{R} = +\infty$), while

$$\lim_{\theta\to 0} d(0,\phi) = 2R\varphi(1-\frac{\varphi}{2\pi}).$$

This is the arc length of the curve $r(\varphi)$, solution of

$$r^2 + \dot{r}^2 = (1-\frac{\varphi}{\pi})^2. \tag{140}$$

(140) has no global solution. Gluing the solution of $\dot{r} = \sqrt{(1-\frac{\varphi}{\pi})^2 - r^2}$ on $[\pi, 2\pi]$ with the solution of $\dot{r} = -\sqrt{(1-\frac{\varphi}{\pi})^2 - r^2}$ on $[0,\pi]$ with initial condition $r(\pi) = 0$, one obtains that at the limit $\theta \to 1$ the base S_ξ, seen for ξ, has the shape of a heart (figure 8). Hence, still from the intrinsic point of view developed from S_x, θ is a deformation parameter for the base of P from an infinite cardioid to a heart. The shape of S_ξ for intermediate values of θ is deserving of further study.

VII Conclusion and outlook

The 2-torus $\mathbb{T}_\xi$ inherits from noncommutative geometry a metric smoother than the Euclidean one (the associated distance function is smooth at the cut locus). It gives to both the fiber and the base the shape of a cardioid or a heart. Such a "smooth cardio-torus" (shall we denote it $\heartsuit_\xi$?) offers a concrete example in which the distance (3) is "truly" noncommutative, in the sense that is not a Riemannian geodesic distance (as in the commutative case), nor a combination of the latter with a discrete space (as in the two-sheet model), not even the Carnot-Carathéodory one. The noncommutative distance combines some aspects of the Euclidean metric on the torus (preservation of the fiber structure) with some aspects of the Sub-Riemannian metric (dependance on the connection).

From a geometrical point of view several questions remain to be studied: what is the metric when both the scalar and the gauge fluctuations are non-zero ? How to extend

the present result to manifolds other than S^1 ? In particular it could be interesting to separate in the holonomy conditions the role of the curvature from the role of the non-connectedness. For instance could it be that, in a certain "local" sense, d equals d_H ? Let us also underline that the present work is intended to be the first step in the computation of the metric aspect of the noncommutative torus where the bundle of pure states P is no longer trivial.

From a physics point of view, it would be interesting to reexamine in the light of the present results some interpretations that were given to sub-Riemannian-geodesics as effective trajectories of particles (Wong's equations). This should be the object of further work.

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